MTH 166

Lecture-29

Revision of Unit-5 and MCQ Practice

Level Surfaces:

Let $f(x, y, z)$ be a given scalar surface.

Level surfaces corresponding to this $f(x, y, z)$ are given by:

$$f(x, y, z) = c \qquad (1)$$

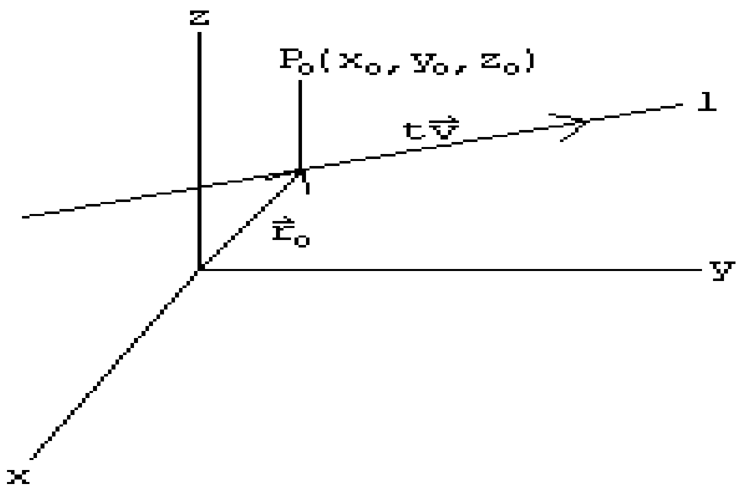
Infact, this equation (1) gives family of surfaces that never intersect with each other

For different values of constant c, we get different members of this family of level surfaces.

Parametric Equation of a Straight line passing through a point and having a given direction:

Let given point is $P_0(x_0, y_0, z_0)$ and the given direction be $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

Then Parametric Equation of a Straight line passing through point P_0 and having given direction $\vec{v}$ is given by:



$\overrightarrow{r(t)} = \overrightarrow{P_0} + t\,\vec{v}$ where t is parameter

$\Rightarrow \overrightarrow{r(t)} = (x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}) + t(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k})$

$\Rightarrow \overrightarrow{r(t)} = (x_0 + tv_1)\hat{\imath} + (y_0 + tv_2)\hat{\jmath} + (z_0 + tv_3)\hat{k}$

Length of Space Curve:

Let the curve C be represented in the parametric form as:

$$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \quad a \leq t \leq b$$

Then, length of curve C is given by:

$$l = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Motion of a body or a particle

Let position vector of a particle be given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Then, we can calculate following quantities:

1. Velocity, $\overrightarrow{V(t)} = \frac{d}{dt}\left(\overrightarrow{r(t)}\right)$

2. Speed, $S = \left|\overrightarrow{V(t)}\right|$ (Magnitude of vector $\overrightarrow{V(t)}$)

3. Acceleration, $\overrightarrow{a(t)} = \frac{d}{dt}\left(\overrightarrow{V(t)}\right)$

Gradient of a scalar field

Let $f(x, y, z)$ be a given scalar surface.

Let us consider a vector differential operator called as **Del/Nabla** defined as:

$$\vec{\nabla} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

Then gradient of scalar field f is given by:

$$\text{grad}(f) = \vec{\nabla} f = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(x, y, z)$$

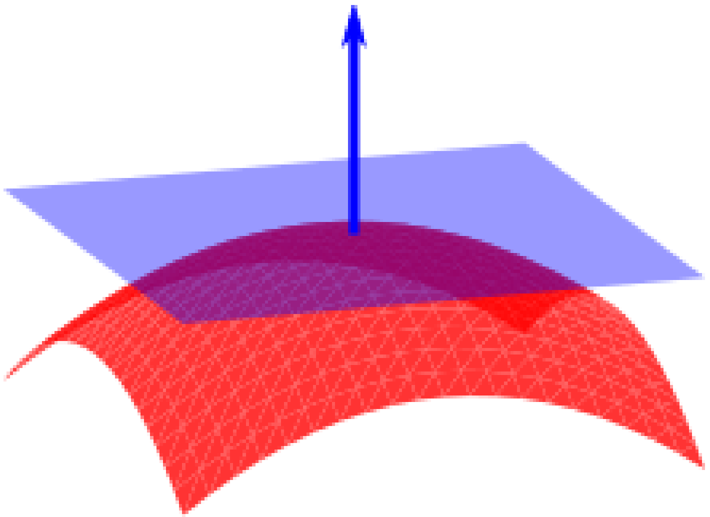
$$= \left(\hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}\right).$$

Normal Vector to a Scalar Field Surface

Geometrically, Gradient of a scalar field represents a vector normal to the surface,

Let $f(x, y, z)$ be a given scalar surface.



Let $\vec{\nabla} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

Then vector normal to the surface f is given by:

Normal vector, $\vec{n} = \text{grad}(f) = \vec{\nabla} f = \left(f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}\right)$

Unit normal vector, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$

Angle between two Scalar Surface

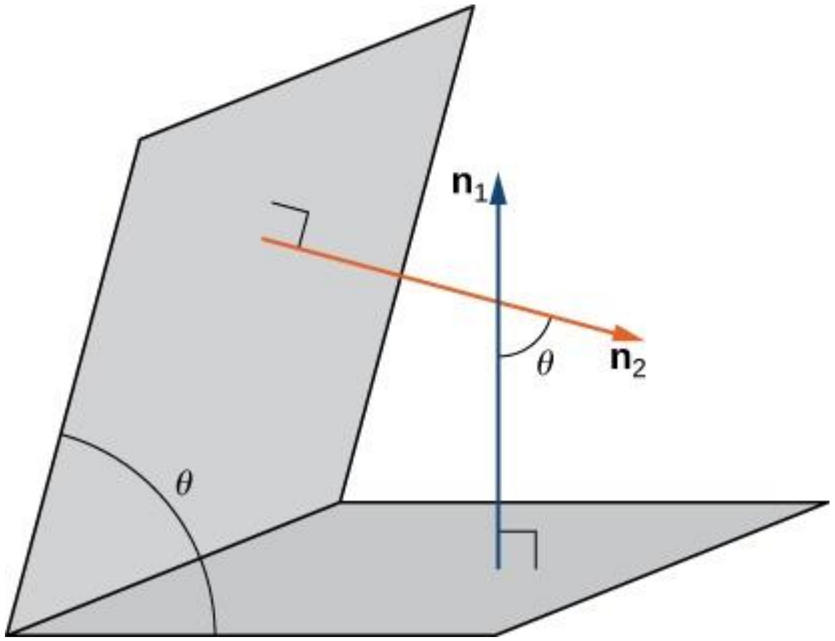
Angle between two surfaces is equal to the angle between their normal.

Let f and g be two given scalar surfaces. Let $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ be the vectors normal to the surfaces f and g respectively.

Then angle between surfaces f and g is given by:

$$\cos\theta = [\widehat{n_1}.\widehat{n_2}] = \left[\frac{\overrightarrow{n_1}}{|\overrightarrow{n_1}|}.\frac{\overrightarrow{n_2}}{|\overrightarrow{n_2}|}\right]$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{\vec{\nabla} f}{|\vec{\nabla} f|}.\frac{\vec{\nabla} g}{|\vec{\nabla} g|}\right]$$



Directional derivative of a scalar field

Let $f(x, y, z)$ be a given scalar surface.

Let the given direction be $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

Then directional derivative of f in the direction of vector $\vec{b}$ is given by:

$$D_{\vec{b}}(f) = \vec{\nabla}f.\,\hat{b} = \vec{\nabla}f.\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}\right).\left(\frac{b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}}{\sqrt{(b_1)^2 + (b_2)^2 + (b_3)^2}}\right)$$

$$= \frac{f_x b_1 + f_y b_2 + f_z b_3}{\sqrt{(b_1)^2 + (b_2)^2 + (b_3)^2}}$$

Note:

1. Maximum rate of increase (Minimum rate of decrease) $= \left|\vec{\nabla} f\right|$

It occurs in its own direction.

2. Minimum rate of increase (Maximum rate of decrease) $= -\left|\vec{\nabla} f\right|$

It occurs in its opposite direction.

Let the given vector field be $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

Let us consider a vector differential operator $\vec{\nabla} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

<u>Divergence:</u> It is dot product between $\vec{\nabla}$ and $\vec{v}$ and is given by:

$$\mathrm{div}(\vec{v}) = \vec{\nabla}.\vec{v} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \qquad (\because \hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1)$$

Divergence is always a scalar quantity (Having no direction).

<u>Note:</u> If $\mathbf{div}(\vec{\boldsymbol{v}}) = \mathbf{0}$, then vector $\vec{v}$ is called **Solenoidal** or **Incompressible**.

That means the field has no sources and no sinks.

Curl: It is cross product between $\vec{\nabla}$ and $\vec{v}$ and is given by:

$$\text{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Curl is always a vector quantity.

Note: If $\mathbf{curl}(\vec{\boldsymbol{v}}) = \vec{\mathbf{0}}$, then vector $\vec{v}$ is called **Irrotational** or **Conservative**.

Two Important Properties:

1. div(curl($\vec{v}$)) = 0 (Always)

2. curl(grad (f)) = $\vec{0}$ (Always)

Important Results to Remember

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ (A Position vector)

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ (A Constant vector)

Let $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ (A vector field)

Let $\vec{\nabla} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$ (A del operator)

Let f and g be two given scalar surfaces.

The following equalities are always true:

1. div $(\vec{r}) = 3$

2. curl $(\vec{r}) = \vec{0}$

3. div (curl($\vec{v}$)) = 0

4. curl (grad (f)) = $\vec{0}$

5. grad $(\vec{a}.\vec{r}) = \vec{a}$

6. div$(\vec{a} \times \vec{r}) = 0$

7. curl $(\vec{a} \times \vec{r}) = 2\vec{a}$

8. div $[(\vec{a}.\vec{r})\vec{r}] = 4(\vec{a}.\vec{r})$

9. div $[(\vec{r}.\vec{r})\vec{a}] = 2(\vec{a}.\vec{r})$

10. div$\left(\frac{\vec{r}}{r^3}\right) = 0$

***** If $\vec{E}$ and $\vec{H}$ are irrotational, then $\vec{E} \times \vec{H}$ is solenoidal

MCQ Practice Questions

In next section we will be discussing some MCQ questions from previous years question papers.

Q1. Parametric equation of circle $x^2 + y^2 = 16$ is:

(A) $x = 4\cos t, \quad y = 4\sin t$ **(B)** $x = 4\cos t, \quad y = 4\sin t$

(C) $x = 16\cos t, \quad y = 16\sin t$ **(D)** None of these

Sol. Look at option (A): $x = 4\cos t, \quad y = 4\sin t$

So, $x^2 + y^2 = (4\cos t)^2 + (4\sin t)^2$

$= 16(cos^2 t + sin^2 t)$

$= 16$

Hence, option **(A)** is correct option.

Q2. Level surfaces of the scalar function $f = x + y + z$ is:

(A) $x + y + z = c$

(B) $x^2 + y^2 + z^2 = c$

(C) $f - x = y + z$

(D) None of these

Sol. The given scalar function is:

$f = x + y + z$

Level surfaces are given by: $f = c$

$\Rightarrow x + y + z = c$

Which is a family of Parallel planes.

Hence, option **(A)** is correct option.

Q3. The given vector field $\vec{v} = 3x^2y^2z^4\hat{\imath} + 2x^3yz^4\hat{\jmath} + 4x^3y^2z^3\hat{k}$ represents the velocity of fluid in a medium, then fluid is:

(A) Incompressible

(B) Solenoidal

(C) Irrotational

(D) None of these

Sol. $\operatorname{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^2z^4 & 2x^3yz^4 & 4x^3y^2z^3 \end{vmatrix}$

$= \hat{\imath}(8x^3yz^3 - 8x^3yz^3) - \hat{\jmath}(12x^2y^2z^3 - 12x^2y^2z^3) + \hat{k}(6x^2yz^4 - 6x^2yz^4)$

$= 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \vec{0}$. So, the given vector $\vec{v}$ is Irrotational.

Hence, option **(C)** is correct option.

Q4. Let $f = x\sin(x + y + z)$**,** then the value of curl (grad f) is

(A) $\sin(x + y + z)$

(B) $x\cos(x + y + z)$

(C) 0

(D) None of these

Sol. Whatever may be the value of grad f, the standard rule says:

curl (grad f)$= \vec{0}$.

Hence, option **(C)** is correct option.

Q5. The length of the curve $\vec{r} = 4\cos t\,\hat{\imath} + 4\sin t\,\hat{\jmath}, 0 \le t \le 2\pi$ is:

(A) 4π

(B) 8π

(C) 16π

(D) None of these

Sol. Comparing with: $\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath}, \quad a \le t \le b$

$x(t) = 4\cos t \quad \Rightarrow \frac{dx}{dt} = -4\sin t$ and $y(t) = 4\sin t \quad \Rightarrow \frac{dy}{dt} = 4\cos t$

Length of curve C is given by: $l = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\Rightarrow l = \int_{t=0}^{t=2\pi} \sqrt{(-4\sin t)^2 + (4\cos t)^2}\, dt = \int_{t=0}^{t=2\pi} \sqrt{16}\, dt = 4(2\pi - 0) = 8\pi$$

Hence, option **(B)** is correct option.

Q6. If $\vec{F}$ is the velocity of fluid in a medium, then fluid is said to be Incompressible if:

(A) curl $\left(\vec{F}\right) = 0$

(B) div $(\vec{F}) = 0$

(C) grad $(|\vec{F}|) = 0$

(D) None of these

Sol. If $\mathbf{div}\left(\vec{\boldsymbol{F}}\right) = \mathbf{0}$, then vector $\vec{v}$ is called **Solenoidal** or **Incompressible**.

That means the field has no sources and no sinks.

Hence, option **(B)** is correct option.

Q7. Curl of a vector point field $\vec{r} = x\hat{\imath} + y\hat{\jmath} + \mathrm{z}\hat{k}$ is:

(A) 0 **(B)** 1

(C) 3 **(D)** 2

Sol. Here curl $(\vec{r}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = 0$

Hence, option **(A)** is correct option.

Q8. Divergence of a vector point field $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is:

(A) 0 **(B)** 1

(C) 3 **(D)** 2

Sol. Comparing with: $\vec{r} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

$v_1 = x, \quad v_2 = y, \quad v_3 = z$

$$\Rightarrow \frac{\partial v_1}{\partial x} = 1, \quad \frac{\partial v_2}{\partial y} = 1, \quad \frac{\partial v_3}{\partial z} = 1$$

$$\text{div}(\vec{v}) = \vec{\nabla}.\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 1 + 1 + 1 = 3$$

Hence, option **(C)** is correct option.

Q9. The normal vector to the curve $y^2 = 16x$ at $(4,8)$ is:

(A) $16(\hat{\imath} + \hat{\jmath})$

(B) $16(\hat{\imath} - \hat{\jmath})$

(C) $\frac{(\hat{\imath}-\hat{\jmath})}{\sqrt{2}}$

(D) $\frac{(\hat{\imath}+\hat{\jmath})}{\sqrt{2}}$

Sol. Let $f(x,y,z) = 16x - y^2$

$\Rightarrow f_x = 16, \quad f_y = -2y$

Then gradient of scalar field f is given by:

$\text{grad}(f) = \vec{\nabla} f = \left(f_x\hat{\imath} + f_y\hat{\jmath}\right) = 16\hat{\imath} - 2y\hat{\jmath}$

At point $(4,8)$: $\vec{\nabla} f = 16\hat{\imath} - 16\hat{\jmath}$

Normal vector, $\vec{n} = \vec{\nabla} f = 16\hat{\imath} - 16\hat{\jmath}$

Unit normal vector, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{16\hat{\imath}-16\hat{\jmath}}{\sqrt{(16)^2+(16)^2}} = \frac{16(\hat{\imath}-\hat{\jmath})}{16\sqrt{1+1}} = \frac{(\hat{\imath}-\hat{\jmath})}{\sqrt{2}}$ **Option (C)**

Q10. The gradient of scalar function $(x^3 - 3x^2y^2 + y^3)$ at point $(1,2)$ is:

(A) $-21\hat{\imath}$

(B) $-21\hat{\jmath}$

(C) 0

(D) None of these

Sol. Let $f(x,y,z) = (x^3 - 3x^2y^2 + y^3)$

$\Rightarrow f_x = (3x^2 - 6xy^2), f_y = (3y^2 - 6x^2y)$

Then gradient of scalar field f is given by:

$\text{grad}(f) = \vec{\nabla}f = \left(f_x\hat{\imath} + f_y\hat{\jmath}\right) = (3x^2 - 6xy^2)\hat{\imath} + (3y^2 - 6x^2y)\hat{\jmath}$

At point $(1,2)$:

$\text{grad}(f) = \vec{\nabla}f = (3 - 24)\hat{\imath} + (12 - 12)\hat{\jmath} = -21\hat{\imath}$

Hence, option **(A)** is correct option.

Q11. The parametric representation of the curve $x + y + z = 3, y - z = 0$ is:

(A) $x = 3 - 2t, \mathrm{y} = \mathrm{t}, \mathrm{z} = \mathrm{t}$

(B) $x = 0, y = 1, z = 1$

(C) $x = 3 - 2t, \mathrm{y} = \mathrm{t}, \mathrm{z} = \mathrm{t}$

(D) $3x = y = z$

Sol. Let $z = t$ where t is a parameter

$\Rightarrow y = t \qquad (\because y = z)$

also $x + y + z = 3$

$\Rightarrow x = 3 - y - z = 3 - t - t$

$\Rightarrow x = 3 - 2t$

Hence, option **(A)** is correct option.

Q12. The parametric representation of the curve $y = 3x + 5$ is:

(A) $x = 3 - 2t, y = t$

(B) $x = t, y = 5t + 3$

(C) $x = 1, y = 8$

(D) None of these

Sol. Let $x = t$ where t is a parameter

$\Rightarrow y = 3t + 5$

Let $t = 1$

$\Rightarrow x = 1$ and $y = 3(1) + 5 = 8$

Hence, option **(C)** is correct option.